GES Attitude Observer with Single Vector Observations [⋆]

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Abstract

This paper proposes a globally exponentially stable (GES) observer for attitude estimation based on a single time-varying reference vector, in inertial coordinates, and corresponding vector, in body-fixed coordinates, in addition to angular velocity readings. The proposed solution is computationally efficient and, in spite of the fact that the observer does not evolve on the Special Orthogonal Group $SO(3)$, an explicit solution on $SO(3)$ is also provided, whose error is shown to converge exponentially fast to zero for all initial conditions. The distinct roles of the inertial and the corresponding body-fixed vectors on the observability of the system are also examined and simulation results are shown that illustrate the performance of the proposed attitude observer in the presence of low-grade sensor specifications.

Key words: Attitude algorithms; state observers; time-varying systems; estimation algorithms; navigation systems.

1. Introduction

Attitude estimation has been a hot topic of research in the past decades, see e.g. [1, 2, 3, 4, 5, 6, 7, 8] and references therein. The reader is referred to [9] for a survey on the topic. However, only recently has attitude estimation been studied based on time-varying reference vectors and, in particular, single vector observations, see [10], [11], [12], and [13]. In [14] an explicit solution on $SO(3)$ is proposed with almost globally asymptotically stable error dynamics.

This paper presents a novel theoretical attitude estimation framework based on single vector observations. Applications include, e.g., attitude estimation of unmanned vehicles that depend on electromagnetic or acoustic feedback of direction vectors of known landmarks. For space applications, it is interesting e.g. in attitude estimation in orbits with gravity gradient effects or even using magnetometers and sun sensor readings, as the corresponding inertial vectors are slowly time-varying. Alternative applications include sensor calibration, see e.g. [10] and [11]. Dual to the topic of attitude estimation is attitude stabilization, see e.g. [15], [16], and [17] and references therein. An interesting separation principle can be found in [18] and [19].

The main contribution of this paper is the development of a novel attitude observer based on single vector observations with globally exponentially stable error dynamics. Central to the observer design is the construction of a set of auxiliary reference vectors (and corresponding vectors in body-fixed coordinates) and the derivation of sufficient observability conditions, which result in appropriate persistent excitation conditions that also allow for norm changes, including null vectors for some time. The proposed design is computationally efficient and the stability analysis builds on well-established Lyapunov results and linear systems theory. Unlike prior contributions that resort to local coordinates, the unit quaternion, or Lie group techniques, the approach followed in this paper consists in the embedding of $SO(3)$ into $\mathbb{R}^9$, considering the problem in an unconstrained fashion. As such, widely known issues such as singularities, unwinding phenomena, slow convergence near unstable equilibium points or topological limitations for achieving global stabilization using smooth feedback on $SO(3)$ do not apply, see [20], [17], [15], [21], and references therein. As the observers do not explicitly evolve on $SO(3)$, an explicit solution on $SO(3)$ is also provided resorting to a projection operator, an approach that has been employed

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previously in the design of interpolation methods on $SE(3)$, see [22]. The error of this solution is shown to converge exponentially fast to zero for all initial conditions.

The paper is organized as follows. The attitude kinematics and some preliminary definitions are given in Section 2, while the problem addressed in the paper is described in Section 3. The design and stability analysis of the attitude observer is presented Section 4. In addition, the roles of the inertial and body-fixed vectors are discussed, as well as some refinements to the proposed solution. Simulation results that illustrate the achievable performance are presented in Section 5 and, finally, Section 6 summarizes the main contributions and conclusions of the paper.

1.1. *Notation*

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. A block diagonal matrix is represented as $\mathrm{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_\mathrm{n})$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \times \mathbf{y}$ represents the cross product.

2. Preliminaries

Let $\{I\}$ be an inertial reference frame, $\{B\}$ a body-fixed reference frame, and $\mathbf{R}(t) \in SO(3)$ the rotation matrix from $\{B\}$ to $\{I\}$. The attitude kinematics are given by $\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}\left[\boldsymbol{\omega}(t)\right]$, where $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ is the angular velocity of $\{B\}$, expressed in $\{B\}$, and $\mathbf{S}\left(.\right)$ is the skew-symmetric matrix such that $\mathbf{S}\left(\boldsymbol{x}\right)\boldsymbol{y} = \boldsymbol{x} \times \boldsymbol{y}$. The angular velocity is assumed to be a continuous bounded signal, available for observer design purposes.

The following definitions are useful in the sequel.

Definition 1 *A continuous norm-bounded vector* $\mathbf{a}(t) \in \mathbb{R}^3$ *is called persistently non-constant if there exist* $\alpha > 0$, $\epsilon > 0$, *and* $\delta > 0$ *such that, for all* $t \geq t_0$ *and* $\mathbf{d} \in \mathbb{R}^3$, $\|\mathbf{d}\| = 1$, *it is true that* $\|\mathbf{a}\left(t\right)\| > \epsilon$ *and*

$$\int_t^{t+\delta} \|\mathbf{a}\left(\sigma\right) \times \mathbf{d}\| \, d\sigma \geq \alpha.$$

Definition 2 *A set of* N *piecewise continuous norm-bounded vectors* $\mathcal{A} = \left\{\mathbf{a}_i(t) \in \mathbb{R}^3, i = 1, \ldots, N\right\}$ *is called persistently non-collinear if there exist* $\alpha > 0$ *and* $\delta > 0$ *such that, for all* $t \geq t_0$, *there exist* l *and* m *such that*

$$\int_t^{t+\delta} \|\mathbf{a}_l\left(\sigma\right) \times \mathbf{a}_m\left(\sigma\right)\| \, d\sigma \geq \alpha.$$

Definition 3 *A set of* N *piecewise continuous norm-bounded vectors* $\mathcal{A} = \left\{\mathbf{a}_i(t) \in \mathbb{R}^3, i = 1, \ldots, N\right\}$ *is called persistently non-planar if there exist* $\alpha > 0$ *and* $\delta > 0$ *such that, for all* $t \geq t_0$ *and* $\mathbf{d} \in \mathbb{R}^3$, $\|\mathbf{d}\| = 1$, *there exist* l, m, *and* n *such that*

$$\int_t^{t+\delta} \|\mathbf{M}\left(\sigma\right)\mathbf{d}\| \, d\sigma \geq \alpha,$$

where $\mathbf{M}(t) := \begin{bmatrix} \mathbf{a}_l(t) & \mathbf{a}_m(t) & \mathbf{a}_n(t) \end{bmatrix}^T \in \mathbb{R}^{3\times 3}$.

3. Problem statement

Consider a persistently non-constant reference vector $\mathbf{r}_1(t) \in \mathbb{R}^3$, expressed in inertial coordinates, and the corresponding vector $\mathbf{v}_1(t) \in \mathbb{R}^3$, expressed in body-fixed coordinates, that satisfies

$$\mathbf{r}_1(t) = \mathbf{R}(t)\mathbf{v}_1(t). \tag{1}$$

Suppose that the angular velocity $\boldsymbol{\omega}(t)$ is continuous and bounded. The problem of attitude estimation considered in the paper is that of designing an observer for the rotation matrix $\mathbf{R}(t)$ with globally exponentially stable error dynamics based on $\mathbf{v}_1(t)$, $\mathbf{r}_1(t)$, and $\boldsymbol{\omega}(t)$.

4. Observer design and stability analysis

The attitude observer proposed in the paper follows by embedding $SO(3)$ in $\mathbb{R}^9$, discarding the topological structure of the Special Orthogonal Group $SO(3)$. In order to simplify the derivation, consider a column stacking of $\mathbf{R}(t)$ given by $\boldsymbol{x}(t) = \begin{bmatrix} \mathbf{z}_1^T(t) & \mathbf{z}_2^T(t) & \mathbf{z}_3^T(t) \end{bmatrix} \in \mathbb{R}^9$, where

$$\mathbf{R}(t) = \begin{bmatrix} \mathbf{z}_1^T(t) \\ \mathbf{z}_2^T(t) \\ \mathbf{z}_3^T(t) \end{bmatrix}, \ \mathbf{z}_i(t) \in \mathbb{R}^3, \ i = 1, \ldots, 3.$$

It is straightforward to show that $\dot{\boldsymbol{x}}(t) = -\mathbf{S}_3\left[\boldsymbol{\omega}(t)\right]\boldsymbol{x}(t)$, where $\mathbf{S}_3\left(\mathbf{x}\right) := \mathrm{diag}\left(\mathbf{S}\left(\mathbf{x}\right), \mathbf{S}\left(\mathbf{x}\right), \mathbf{S}\left(\mathbf{x}\right)\right) \in \mathbb{R}^{9\times 9}$, $\mathbf{x} \in \mathbb{R}^3$. An attitude observer for a persistently non-planar set of reference vectors is derived in Section 4.1. The observer for single vectors follows in Section 4.2 by constructing a persistently non-planar set of reference vectors in inertial coordinates (and corresponding body-fixed vectors) from a single persistently non-constant reference vector. The solutions provided by these observers converge asymptotically to $SO(3)$ but do not necessarily evolve on $SO(3)$. In Section 4.3 an explicit solution on $SO(3)$ is provided and examined, while the distinct roles of the reference vectors and body-fixed vectors are analysed in Section 4.4. Finally, in Section 4.5, additional discussion on the proposed observers is offered.

4.1. *Persistently non-planar reference vectors*

Consider a set of vectors $\mathcal{V} = \left\{\mathbf{v}_i(t) \in \mathbb{R}^3, i = 1, \ldots, N\right\}$ expressed in body-fixed coordinates associated with a set of vectors $\mathcal{R} = \left\{\mathbf{r}_i(t) \in \mathbb{R}^3, i = 1, \ldots, N\right\}$ expressed in inertial coordinates, such that $\mathbf{r}_i(t) = \mathbf{R}(t)\mathbf{v}_i(t), \ i = 1, \ldots, N$. Then, it is straightforward to show that $\mathbf{v}(t) = \mathbf{C}(t)\boldsymbol{x}(t)$, where $\mathbf{v}(t) := \begin{bmatrix} \mathbf{v}_1^T(t) & \ldots & \mathbf{v}_N^T(t) \end{bmatrix}^T \in \mathbb{R}^{3N}$ and

$$\mathbf{C}(t) := \begin{bmatrix} r_{11}(t)\mathbf{I}_3 & r_{12}(t)\mathbf{I}_3 & r_{13}(t)\mathbf{I}_3 \\ & \vdots & \\ r_{N1}(t)\mathbf{I}_3 & r_{N2}(t)\mathbf{I}_3 & r_{N3}(t)\mathbf{I}_3 \end{bmatrix} \in \mathbb{R}^{3N\times 9},$$

with $\mathbf{r}_i(t) = [r_{i1}(t)\ r_{i2}(t)\ r_{i3}(t)]^T \in \mathbb{R}^3,\ i = 1, \ldots, N$.

Consider the attitude observer given by

$$\dot{\hat{\boldsymbol{x}}}(t) = -\mathbf{S}_3\left[\boldsymbol{\omega}(t)\right]\hat{\boldsymbol{x}}(t) + \mathbf{C}^T(t)\mathbf{Q}\left[\mathbf{v}(t) - \mathbf{C}(t)\hat{\boldsymbol{x}}(t)\right], \quad (2)$$

where $\mathbf{Q} = \mathbf{Q}^T \in \mathbb{R}^{3N\times 3N}$ is a positive definite matrix, and define the error variable $\tilde{\boldsymbol{x}}(t) := \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$. Then, the observer error dynamics are given by

$$\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{\mathcal{A}}(t)\tilde{\boldsymbol{x}}(t), \quad (3)$$

where $\boldsymbol{\mathcal{A}}(t) := -\left(\mathbf{S}_3\left[\boldsymbol{\omega}(t)\right] + \mathbf{C}^T(t)\mathbf{Q}\mathbf{C}(t)\right)$.

The following theorem is the main result of this section.

Theorem 4 *Suppose that the set of vectors $\mathcal{R}$ is persistently non-planar and consider the attitude observer* (2), *where $\mathbf{Q} \succ \mathbf{0}$ is a design parameter. Then, the origin of the observer error dynamics* (3) *is a globally exponentially stable equilibrium point.*

PROOF. Let $V(t) := \frac{1}{2}\left\|\tilde{\boldsymbol{x}}(t)\right\|^2$ be a Lyapunov candidate function. As $\mathbf{S}_3\left(.\right)$ is skew-symmetric, it follows that $\dot{V}(t) = -\tilde{\boldsymbol{x}}^T(t)\mathbf{C}^T(t)\mathbf{Q}\mathbf{C}(t)\tilde{\boldsymbol{x}}(t)$, which can be written as $\dot{V}(t) = -\tilde{\boldsymbol{x}}^T(t)\boldsymbol{\mathcal{C}}^T(t)\boldsymbol{\mathcal{C}}(t)\tilde{\boldsymbol{x}}(t)$, where $\boldsymbol{\mathcal{C}}(t) := \mathbf{Q}^{\frac{1}{2}}\mathbf{C}(t)$. Clearly, $\frac{1}{2}\left\|\tilde{\boldsymbol{x}}(t)\right\|^2 \leq V(t) \leq \frac{1}{2}\left\|\tilde{\boldsymbol{x}}(t)\right\|^2$ and $\dot{V}(t) \leq 0$. If, in addition, the pair $(\boldsymbol{\mathcal{A}}(t),\boldsymbol{\mathcal{C}}(t))$ is uniformly completely observable, then the origin of the linear time-varying system (3) is a globally exponentially stable equilibrium point, see [23, Example 8.11]. The remainder of the proof amounts to show that this is the case. For any piecewise continuous, bounded matrix $\mathbf{K}(t)$, of compatible dimensions, uniform complete observability of the pair $(\boldsymbol{\mathcal{A}}(t),\boldsymbol{\mathcal{C}}(t))$ is equivalent to uniform complete observability of the pair $(\mathbf{A}(t),\boldsymbol{\mathcal{C}}(t))$, with $\mathbf{A}(t) := \boldsymbol{\mathcal{A}}(t) - \mathbf{K}(t)\boldsymbol{\mathcal{C}}(t)$, see [24, Lemma 4.8.1]. Now, notice that, with $\mathbf{K}(t) = -\boldsymbol{\mathcal{C}}^T(t)$, it follows that $\mathbf{A}(t) = -\mathbf{S}_3\left[\boldsymbol{\omega}(t)\right]$. Therefore, it remains to show that there exist positive constants ϵ_1, ϵ_2, and δ such that

$$\epsilon_1\mathbf{I} \preceq \boldsymbol{\mathcal{W}}\left(t, t+\delta\right) \preceq \epsilon_2\mathbf{I} \quad (4)$$

for all $t \geq t_0$, where $\boldsymbol{\mathcal{W}}\left(t_0, t_f\right)$ is the observability Gramian associated with the pair $(\mathbf{A}(t),\boldsymbol{\mathcal{C}}(t))$ on $[t_0, t_f]$. Since the entries of both $\mathbf{A}(t)$ and $\boldsymbol{\mathcal{C}}(t)$ are continuous and bounded, it is trivial to show that, for any positive δ, there always exists a positive ϵ_2, depending on δ, such that the right side of (4) is verified. Therefore, it remains to show that there exist positive constants δ and ϵ_1 such that the left side of (4) holds. It is straightforward to show that the transition matrix associated with $\mathbf{A}(t)$ is given by

$$\boldsymbol{\phi}\left(t, t_0\right) = \operatorname{diag}\left(\mathbf{R}^{\mathrm{T}}(\mathrm{t})\mathbf{R}\left(\mathrm{t}_0\right), \mathbf{R}^{\mathrm{T}}(\mathrm{t})\mathbf{R}\left(\mathrm{t}_0\right), \mathbf{R}^{\mathrm{T}}(\mathrm{t})\mathbf{R}\left(\mathrm{t}_0\right)\right).$$

Let $\mathbf{d} = \begin{bmatrix} \mathbf{d}_1^T & \mathbf{d}_2^T & \mathbf{d}_3^T \end{bmatrix}^T \in \mathbb{R}^9$, $\mathbf{d}_i \in \mathbb{R}^3$, $i = 1, 2, 3$, be a unit vector. Then,

$$\mathbf{d}^T\boldsymbol{\mathcal{W}}\left(t, t+\delta\right)\mathbf{d} = \int_t^{t+\delta} \mathbf{d}^T\boldsymbol{\phi}^T(\tau, t)\,\mathbf{C}^T(\tau)\mathbf{Q}\mathbf{C}(\tau)\boldsymbol{\phi}\left(\tau, t\right)\mathbf{d}d\tau$$

for all $t \geq t_0$ and $\delta > 0$. As $\mathbf{Q}$ is positive definite, it follows, using the Rayleigh-Ritz inequality, that

$$\mathbf{d}^T\boldsymbol{\mathcal{W}}\left(t, t+\delta\right)\mathbf{d} \geq c_1 \int_t^{t+\delta} \left\|\mathbf{C}(\tau)\boldsymbol{\phi}\left(\tau, t\right)\mathbf{d}\right\|^2 d\tau, \quad (5)$$

where c_1 is the minimum eigenvalue of $\mathbf{Q}$, which is strictly positive. It is straightforward to show that

$$\mathbf{C}(\tau)\boldsymbol{\phi}\left(\tau, t\right)\mathbf{d} = \begin{bmatrix} \mathbf{R}^T\left(\tau\right)\mathbf{R}\left(t\right)\left[\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3\right]\mathbf{r}_1(\tau) \\ \vdots \\ \mathbf{R}^T\left(\tau\right)\mathbf{R}\left(t\right)\left[\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3\right]\mathbf{r}_N(\tau) \end{bmatrix}.$$

As $\mathbf{R}^T\left(\tau\right)\mathbf{R}\left(t\right) \in SO(3)$, it follows that

$$\left\|\mathbf{C}(\tau)\boldsymbol{\phi}\left(\tau, t\right)\mathbf{d}\right\| = \left\|\begin{bmatrix} \left[\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3\right]\mathbf{r}_1(\tau) \\ \vdots \\ \left[\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3\right]\mathbf{r}_N(\tau) \end{bmatrix}\right\|$$

and therefore it is possible to rewrite (5) as

$$\mathbf{d}^T\boldsymbol{\mathcal{W}}\left(t, t+\delta\right)\mathbf{d} \geq c_1 \int_t^{t+\delta} \left\|\begin{bmatrix} \left[\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3\right]\mathbf{r}_1(\tau) \\ \vdots \\ \left[\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3\right]\mathbf{r}_N(\tau) \end{bmatrix}\right\|^2 d\tau.$$

As the set of vectors $\mathcal{R}$ is assumed persistently non-planar, it follows, from Lemma 8 (see Appendix A), that there exist positive constants c_2 and δ such that

$$\int_t^{t+\delta} \left\|\begin{bmatrix} \left[\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3\right]\mathbf{r}_1(\tau) \\ \vdots \\ \left[\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3\right]\mathbf{r}_N(\tau) \end{bmatrix}\right\|^2 d\tau \geq c_2$$

for all $t \geq t_0$, which allows to write $\mathbf{d}^T\boldsymbol{\mathcal{W}}\left(t, t+\delta\right)\mathbf{d} \geq \alpha$, for all $t \geq t_0$ and all $\|\mathbf{d}\| = 1$, with $\alpha := c_1 c_2$. Therefore, the pair $(\mathbf{A}(t),\boldsymbol{\mathcal{C}}(t))$ is uniformly completely observable, which concludes the proof. □

4.2. *Persistently non-constant reference vector*

This section presents an attitude observer for a persistently non-constant reference vector. First, the following theorem is introduced, which allows to derive a persistently non-planar set of reference vectors and corresponding vectors in body-fixed coordinates based on a single reference vector and corresponding vector in body-fixed coordinates.

Theorem 5 *Consider a persistently non-constant reference vector $\mathbf{r}_1(t) \in \mathbb{R}^3$ for which there exist $\alpha > 0$ and $\delta > 0$ such that, for all $t \geq t_0$ and $\mathbf{d} \in \mathbb{R}^3$, with $\|\mathbf{d}\| = 1$, it is true that*

$$\int_t^{t+\delta} \left\|\mathbf{r}_1\left(\sigma\right) \times \mathbf{d}\right\| d\sigma \geq \alpha, \quad (6)$$

corresponding to a vector in body-fixed coordinates $\mathbf{v}_1(t) \in \mathbb{R}^3$ such that (1) *holds. Suppose that the angular velocity*

$\boldsymbol{\omega}(t)$ is continuous and bounded. Define the set of reference vectors $\mathcal{R} := \{\mathbf{r}_1(t), \mathbf{r}_2(t), \mathbf{r}_3(t)\}$, where

$$\mathbf{r}_2(t) := \mathbf{r}_1(t_i),\ t_i \le t < t_{i+1},\ i \in \mathbb{N}_0, \tag{7}$$

with $t_i := t_0 + i\delta,\ i \in \mathbb{N}_0$, and

$$\mathbf{r}_3(t) := \mathbf{r}_1(t) \times \mathbf{r}_2(t) \in \mathbb{R}^3. \tag{8}$$

Define also the corresponding set of vectors in body-fixed coordinates $\mathcal{V} := \{\mathbf{v}_1(t), \mathbf{v}_2(t), \mathbf{v}_3(t)\}$, where $\mathbf{v}_2(t)$ is the piecewise continuous vector

$$\begin{cases} \mathbf{v}_2(t_i) := \mathbf{v}_1(t_i) \\ \dot{\mathbf{v}}_2(t) = -\mathbf{S}[\boldsymbol{\omega}(t)]\,\mathbf{v}_2(t),\ t_i \le t < t_{i+1} \end{cases},\ i \in \mathbb{N}_0 \tag{9}$$

and

$$\mathbf{v}_3(t) := \mathbf{v}_1(t) \times \mathbf{v}_2(t) \in \mathbb{R}^3. \tag{10}$$

Then,

i) the set of reference vectors $\mathcal{R}$ is compatible with the set of body-fixed vectors $\mathcal{V}$, i.e.,

$$\mathbf{r}_i(t) = \mathbf{R}(t)\mathbf{v}_i(t),\ i = 1,\ 2,\ 3; \tag{11}$$

and

ii) the set of reference vectors $\mathcal{R}$ is persistently non-planar.

PROOF. First, notice that, by assumption, (11) is verified for $i = 1$. From (9) it is straightforward to show that $\mathbf{v}_2(t) = \mathbf{R}^T(t)\mathbf{R}(t_i)\,\mathbf{v}_2(t_i),\ t_i \le t < t_{i+1},\ i \in \mathbb{N}_0$. By definition, $\mathbf{v}_2(t_i) = \mathbf{v}_1(t_i) = \mathbf{R}^T(t_i)\,\mathbf{r}_1(t_i),\ i \in \mathbb{N}_0$ which allows to write

$$\mathbf{v}_2(t) = \mathbf{R}^T(t)\mathbf{R}(t_i)\,\mathbf{R}^T(t_i)\,\mathbf{r}_1(t_i) = \mathbf{R}^T(t)\mathbf{r}_1(t_i) \tag{12}$$

for all $t_i \le t < t_{i+1},\ i \in \mathbb{N}_0$. Now, substituting (7) in (12) immediately gives that (11) is verified for $i = 2$. As (11) is verified for $i = 1$ and $i = 2$, it is trivial to show, from (8) and (10), that it is also verified for $i = 3$,

$$\mathbf{v}_3(t) = \left[\mathbf{R}^T(t)\mathbf{r}_1(t)\right] \times \left[\mathbf{R}^T(t)\mathbf{r}_2(t)\right] = \mathbf{R}^T(t)\mathbf{r}_3(t),$$

which concludes the first part of the proof. Next, it is shown that the set of vectors $\{\mathbf{r}_1(t), \mathbf{r}_2(t)\}$ is persistently non-collinear. By assumption, there exist positive constants α and δ such that (6) holds. Let $\delta' := 2\delta$. Then,

$$\int_t^{t+\delta'} \|\mathbf{r}_1(\sigma) \times \mathbf{r}_2(\sigma)\|\, d\sigma = \int_{t_i}^{t_{i+1}} \|\mathbf{r}_1(\sigma) \times \mathbf{r}_2(\sigma)\|\, d\sigma$$
$$+ \int_t^{t_i} \|\mathbf{r}_1(\sigma) \times \mathbf{r}_2(\sigma)\|\, d\sigma + \int_{t_{i+1}}^{t+\delta'} \|\mathbf{r}_1(\sigma) \times \mathbf{r}_2(\sigma)\|\, d\sigma \tag{13}$$

for all $t \ge t_0$, where i corresponds to the smallest integer such that $t_i \ge t$. Since all terms are positive, it follows from (13) that

$$\int_t^{t+\delta'} \|\mathbf{r}_1(\sigma) \times \mathbf{r}_2(\sigma)\|\, d\sigma \ge \int_{t_i}^{t_{i+1}} \|\mathbf{r}_1(\sigma) \times \mathbf{r}_2(\sigma)\|\, d\sigma \tag{14}$$

for all $t \ge t_0$. Now, notice that, by definition, $\mathbf{r}_2(t)$ is constant on $[t_i, t_{i+1}[$. Therefore, it is possible to write, from (14),

$$\int_t^{t+\delta'} \|\mathbf{r}_1(\sigma) \times \mathbf{r}_2(\sigma)\|\, d\sigma \ge \int_{t_i}^{t_i+\delta} \|\mathbf{r}_1(\sigma) \times \mathbf{r}_2(t_i)\|\, d\sigma$$
$$\ge \|\mathbf{r}_2(t_i)\| \int_{t_i}^{t_i+\delta} \left\|\mathbf{r}_1(\sigma) \times \frac{\mathbf{r}_2(t_i)}{\|\mathbf{r}_2(t_i)\|}\right\| d\sigma \tag{15}$$

for all $t \ge t_0$. Now, applying (6) to (15) gives

$$\int_t^{t+\delta'} \|\mathbf{r}_1(\sigma) \times \mathbf{r}_2(\sigma)\|\, d\sigma \ge \alpha\, \|\mathbf{r}_2(t_i)\| \tag{16}$$

for all $t \ge t_0$. Substituting (7) in (16), and from the fact that, by assumption, $\mathbf{r}_1(t)$ is persistently non-constant and therefore bounded from below (and above), it follows that the set of reference vectors $\{\mathbf{r}_1(t), \mathbf{r}_2(t)\}$ is persistently non-collinear. Now, notice that the set of vectors $\mathcal{R}$ corresponds to the set of vectors $\{\mathbf{r}_1(t), \mathbf{r}_2(t)\}$, which is persistently non-collinear, augmented with the vector $\mathbf{r}_1(t) \times \mathbf{r}_2(t)$. Therefore, it immediately follows, using Proposition 9 (see Appendix A), that the augmented set $\mathcal{R}$ is persistently non-planar, which concludes the second part of the proof. □

The following theorem is the main result of the paper.

Theorem 6 *Let $\mathbf{r}_1(t) \in \mathbb{R}^3$ be a persistently non-constant vector, expressed in inertial coordinates, and $\mathbf{v}_1(t) \in \mathbb{R}^3$ the corresponding vector expressed in body-fixed coordinates. Consider two sets of vectors $\mathcal{R}$ and $\mathcal{V}$ derived according to Theorem 5, where $\mathcal{R}$ is the set of reference vectors and $\mathcal{V}$ is the corresponding set of vectors expressed in body-fixed coordinates. Further consider the attitude observer (2), where $\mathbf{Q} \succ \mathbf{0}$ is a design parameter. Then, the origin of the observer error dynamics (3) is a globally exponentially stable equilibrium point.*

PROOF. As the augmented set of reference vectors $\mathcal{R}$ was chosen according to Theorem 5, it follows that it is a persistently non-planar set of reference vectors. In addition, the augmented set of body-fixed vectors $\mathcal{V}$ is coherent with the definition of the reference vectors, i.e., (11) is satisfied. Therefore, straightforward application of Theorem 4 yields the desired result. □

4.3. *Solution on $SO(3)$*

The attitude observers previously proposed yield estimates of the rotation matrix $\mathbf{R}(t)$ given by

$$\hat{\mathbf{R}}(t) = \begin{bmatrix} \hat{\mathbf{z}}_1^T(t) \\ \hat{\mathbf{z}}_2^T(t) \\ \hat{\mathbf{z}}_3^T(t) \end{bmatrix},\ \hat{\mathbf{z}}_i(t) \in \mathbb{R}^3,\ i = 1, \ldots, 3,$$

where $\hat{\boldsymbol{x}}(t) = \begin{bmatrix} \hat{\mathbf{z}}_1^T(t) & \hat{\mathbf{z}}_2^T(t) & \hat{\mathbf{z}}_3^T(t) \end{bmatrix}^T \in \mathbb{R}^9$. However, $\hat{\mathbf{R}}(t)$ is not necessarily a rotation matrix as there is nothing in the observer structure imposing the restriction $\hat{\mathbf{R}}(t) \in SO(3)$. In fact, if this restriction is imposed, it is actually impossible to achieve continuous global asymptotic stabilization due to topological limitations, see [20]. Nevertheless, the estimation error of the proposed observer converges globally exponentially fast to zero and therefore the corresponding rotation matrix restrictions are verified asymptotically. When the observer error is sufficiently small, one orthogonalization cycle suffices, as given by

$$\hat{\mathbf{R}}_o(t) = \frac{1}{2}\left(\hat{\mathbf{R}}(t) + \left[\hat{\mathbf{R}}^T(t)\right]^{-1}\right),$$

to obtain an estimate sufficiently close to an element of $SO(3)$, see [25]. In spite of the fact that orthogonalization cycles are an extremely efficient method to obtain an estimate of the rotation matrix that is very close to $SO(3)$, it may happen that an explicit solution on SO(3) is required. This is established in the following theorem.

Theorem 7 *Consider the estimate $\hat{\mathbf{R}}(t)$ obtained from the attitude observer* (2)*, under the conditions of Theorem 4 (or Theorem 6), with GES error dynamics. Further suppose that the initial estimate satisfies $\hat{\mathbf{R}}(t_0) \in SO(3)$ and define a new attitude estimate $\hat{\mathbf{R}}_f(t)$ of the rotation matrix $\mathbf{R}(t)$ as*

$$\begin{cases} \hat{\mathbf{R}}_f(t) = \arg\min\limits_{\mathbf{X}(t)\in SO(3)} \left\|\mathbf{X}(t) - \hat{\mathbf{R}}(t)\right\|, \left\|\hat{\mathbf{R}}^T(t)\hat{\mathbf{R}}(t) - \mathbf{I}\right\| \leq \epsilon \\ \dot{\hat{\mathbf{R}}}_f(t) = \hat{\mathbf{R}}_f(t)\mathbf{S}\left(\boldsymbol{\omega}(t)\right), \left\|\hat{\mathbf{R}}^T(t)\hat{\mathbf{R}}(t) - \mathbf{I}\right\| > \epsilon \end{cases},$$

where $\epsilon > 0$. Then,

(i) $\hat{\mathbf{R}}_f(t) \in SO(3)$;

(ii) there exists t_s such that $\left\|\hat{\mathbf{R}}^T(t)\hat{\mathbf{R}}(t) - \mathbf{I}\right\| \leq \epsilon$ for all $t \geq t_s$ and therefore $\hat{\mathbf{R}}_f(t)$ corresponds to the projection on $SO(3)$ of $\hat{\mathbf{R}}(t)$ for all $t \geq t_s$; and

(iii) the error $\tilde{\mathbf{R}}_f(t) := \left\|\mathbf{R}(t) - \hat{\mathbf{R}}_f(t)\right\|$ is bounded and $\lim_{t\to\infty}\left\|\tilde{\mathbf{R}}_f(t)\right\| = 0$. Moreover, the convergence is exponentially fast.

PROOF. The first part of the proof, that $\hat{\mathbf{R}}_f(t) \in SO(3)$, follows by construction, which in turn gives that $\left\|\tilde{\mathbf{R}}_f(t)\right\|$ is bounded as $\left\|\tilde{\mathbf{R}}_f(t)\right\| \leq \|\mathbf{R}(t)\| + \left\|\hat{\mathbf{R}}_f(t)\right\| \leq 2$. Define $\tilde{\mathbf{R}}(t) := \mathbf{R}(t) - \hat{\mathbf{R}}(t)$. As $\boldsymbol{x}(t)$ and $\hat{\boldsymbol{x}}(t)$ are column representations of $\mathbf{R}(t)$ and $\hat{\mathbf{R}}(t)$, respectively, and as $\tilde{\boldsymbol{x}}(t)$ converges globally exponentially fast to zero, it follows that $\lim_{t\to\infty}\left\|\tilde{\mathbf{R}}(t)\right\| = 0$. This mean that, for every $\epsilon_1 > 0$, it is possible to choose $t^* \in \mathbb{R}$ such that, for all $t \geq t^*$, it is true that

$$\left\|\mathbf{R}(t) - \hat{\mathbf{R}}(t)\right\| < \epsilon_1, \tag{17}$$

or, equivalently,

$$\left\|\mathbf{R}^T(t)\hat{\mathbf{R}}(t) - \mathbf{I}\right\| < \epsilon_1. \tag{18}$$

Now, notice that

$$\begin{aligned} \hat{\mathbf{R}}^T(t)\hat{\mathbf{R}}(t) - \mathbf{I} &= \mathbf{R}^T(t)\hat{\mathbf{R}}(t) - \mathbf{I} + \left[\hat{\mathbf{R}}(t) - \mathbf{R}(t)\right]^T\hat{\mathbf{R}}(t) \\ &= \mathbf{R}^T(t)\hat{\mathbf{R}}(t) - \mathbf{I} + \tilde{\mathbf{R}}^T(t)\left[\tilde{\mathbf{R}}(t) - \mathbf{R}(t)\right] \end{aligned} \tag{19}$$

and, using simple norm inequalities in (19) allows to write

$$\left\|\hat{\mathbf{R}}^T(t)\hat{\mathbf{R}}(t) - \mathbf{I}\right\| \leq \left\|\mathbf{R}^T(t)\hat{\mathbf{R}}(t) - \mathbf{I}\right\| + \left\|\tilde{\mathbf{R}}(t)\right\|^2 + \left\|\tilde{\mathbf{R}}(t)\right\|. \tag{20}$$

Given $\epsilon > 0$, choose $\epsilon_1 > 0$ such that $\epsilon_1^2 + 2\epsilon_1 \leq \epsilon$. Then, using (17) and (18) in (20) immediately allows to conclude the second part of the theorem. To show that the error $\left\|\tilde{\mathbf{R}}_f(t)\right\|$ converges to zero notice that, for all $t \geq t_s$, $\hat{\mathbf{R}}_f(t)$ corresponds to the projection on $SO(3)$ of $\hat{\mathbf{R}}(t)$ and therefore, for all $t \geq t_s$ and $\mathbf{X}(t) \in SO(3)$, it is true that $\left\|\hat{\mathbf{R}}_f(t) - \hat{\mathbf{R}}(t)\right\| \leq \left\|\mathbf{X}(t) - \hat{\mathbf{R}}(t)\right\|$. In particular, with $\mathbf{X}(t) = \mathbf{R}(t)$ it follows that

$$\left\|\hat{\mathbf{R}}_f(t) - \hat{\mathbf{R}}(t)\right\| \leq \left\|\mathbf{R}(t) - \hat{\mathbf{R}}(t)\right\| \tag{21}$$

for all $t \geq t_s$. Simple norm inequalities allow to write

$$\begin{aligned} \left\|\mathbf{R}(t) - \hat{\mathbf{R}}_f(t)\right\| &\leq \left\|\mathbf{R}(t) - \hat{\mathbf{R}}(t) + \hat{\mathbf{R}}(t) - \hat{\mathbf{R}}_f(t)\right\| \\ &\leq \left\|\mathbf{R}(t) - \hat{\mathbf{R}}(t)\right\| + \left\|\hat{\mathbf{R}}_f(t) - \hat{\mathbf{R}}(t)\right\|. \end{aligned} \tag{22}$$

Substituting (21) in (22) gives

$$\left\|\mathbf{R}(t) - \hat{\mathbf{R}}_f(t)\right\| \leq 2\left\|\mathbf{R}(t) - \hat{\mathbf{R}}(t)\right\| \tag{23}$$

for all $t \geq t_s$. As the error of the observer $\left\|\tilde{\mathbf{R}}(t)\right\|$ converges exponentially fast to zero, it follows from (23) that so does $\left\|\mathbf{R}(t) - \hat{\mathbf{R}}_f(t)\right\|$, which concludes the proof. □

Remark 1 *The projection on $SO(3)$ of $\hat{\mathbf{R}}(t)$ is well known and it is readily obtained from the Singular Value Decomposition (SVD) of $\hat{\mathbf{R}}(t)$, see e.g. [22] for more details.*

Remark 2 *Notice that Theorem 7 does not violate any of the results presented in [20] for global asymptotic stabilization on $SO(3)$ as the solution provided by Theorem 7 is not guaranteed to be continuous for $t \leq t_s$. However, for $t > t_s$ the solution is guaranteed to be continuous in the absence of noise. In the presence of noise, appropriate bounds on ϵ, dependent on the noise characteristics, must be imposed in order to guarantee the existence of t_s such that the second result of Theorem 7 holds. This will be detailed in future work.*

4.4. *Roles of the reference and the body-fixed vectors*

This section clarifies the roles of the reference vector in inertial coordinates and the corresponding vector in body-fixed coordinates. First, it is shown that, even though a reference vector may be persistently non-constant, which allows for the application of Theorem 6, the corresponding vector in body-fixed coordinates may not be persistently non-constant. Let $\mathbf{r}_1(t) \in \mathbb{R}^3$ be a persistently non-constant reference vector, in inertial coordinates, and $\mathbf{v}_1(t)$ be the corresponding vector in body-fixed coordinates, which satisfies (1). Let the dynamics of the reference vector be given by $\dot{\mathbf{r}}_1(t) = -\mathbf{S}\left[\boldsymbol{\omega}_r(t)\right]\mathbf{r}_1(t) + u_r(t)\mathbf{r}_1(t)$, where $u_r(t) \in \mathbb{R}$ is a continuous bounded function. Taking the time derivative of $\mathbf{v}_1(t)$ gives $\dot{\mathbf{v}}_1(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{v}_1(t) - \mathbf{R}^T(t)\mathbf{S}\left[\boldsymbol{\omega}_r(t)\right]\mathbf{r}_1(t) + u_r(t)\mathbf{R}^T(t)\mathbf{r}_1(t)$ which, using (1) and the fact that $\mathbf{R}(t)$ is a rotation matrix, together with cross product properties, may be rewritten as

$$\dot{\mathbf{v}}_1(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t) + \mathbf{R}^T(t)\boldsymbol{\omega}_r(t)\right]\mathbf{v}_1(t) + u_r(t)\mathbf{v}_1(t). \quad (24)$$

Now, notice that, with $\boldsymbol{\omega}(t) = -\mathbf{R}^T(t)\boldsymbol{\omega}_r(t)$, (24) reads as $\dot{\mathbf{v}}_1(t) = u_r(t)\mathbf{v}_1(t)$, which means that $\mathbf{v}_1(t)$ has constant direction and therefore it is not persistently non-constant. This shows that it is possible to estimate the attitude even if the vector in body-fixed coordinates is not persistently non-constant.

On the other hand, when the reference vector is not persistently exciting, in the limit situation it has constant direction. It is well known that, with just one constant direction in inertial coordinates, it is impossible to recover the attitude matrix. Nevertheless, the corresponding vector, in body-fixed coordinates, may be persistently exciting, e.g., if $\boldsymbol{\omega}(t) = \boldsymbol{\omega}_v$, where $\boldsymbol{\omega}_v$ is orthogonal to $\mathbf{v}_1(t_0)$. This shows that it may be impossible to estimate the attitude even though the vector observation, in body-fixed coordinates, is persistently non-constant.

The previous discussion clarifies why the inertial vectors are denoted as reference vectors in this paper. Even though it is possible to express alternative conditions such that the observer error dynamics are GES, the observability of the attitude is fundamentally connected to the evolution of the inertial vector. Naturally, the roles of the vectors would differ if the angular velocity of $\{B\}$ with respect to $\{I\}$ was expressed in inertial coordinates.

4.5. *Further discussion*

Lower bound on the norm of the reference vector

In the definition of a persistently non-constant vector, a lower bound is set on the norm of the vector. It turns out that it is possible to generalize the observer design proposed in Section 4.2 to discard this restriction. Essentially, this assumption allows to write (15), where $\|\mathbf{r}_2(t_i)\| = \|\mathbf{r}_1(t_i)\|$ is lower bounded. If this bound was not assumed, it would be possible to have a persistently non-constant vector $\mathbf{r}_1(t)$ in Theorem 5 such that $\mathbf{r}_2(t) = \mathbf{r}_3(t) = \mathbf{0}$ for all t, and therefore the set of vectors $\mathcal{R}$ would not be persistently non-planar. However, an alternate time-varying scheme for the definition of a set of reference vectors is possible where δ is allowed to increase for the update of the reference vector $\mathbf{r}_2(t)$ such that $\mathbf{r}_2(t)$ is not a null vector for all t. The design is trivial and therefore it is omitted.

Direction vs. norm of the reference vector

It is trivially shown that the persistent excitation conditions expressed in the paper are essentially related to the direction of the reference vectors, and the only condition that the norm has to satisfy is not to be convergent to zero, which would lead to the loss of excitation.

Observer design for persistently non-collinear sets of reference vectors

Even though an observer for a persistently non-collinear set of reference vectors is not presented in the paper, this is straightforward and follows by building an augmented set that is persistently non-planar, with the additional vectors resulting from the cross product of existing reference vectors.

Speed vs. performance trade-off

The observer has essentially one tuning knob, the positive definite matrix $\mathbf{Q}$. The larger this is, the faster the error converges to zero. However, as $\mathbf{Q}$ increases, the sensitivity of the estimates to sensor noise increases. Two sets of gains may be employed, one during start-up, and the other during steady-state, to achieve faster convergence if needed while maintaining good steady-state performance.

5. Simulation results

In order to evaluate the performance of the proposed observer, simulations were carried out in a realistic setup environment, considering a single persistently non-constant reference vector, which results in a persistently non-coplanar set of three reference vectors, as given by Theorem 5, where it was set $\delta = 10$ s. The evolution of the reference and body-fixed vectors is depicted in Figs. 1 and 2, respectively. The initial attitude is $\mathbf{R}(0) = \mathbf{I}$. Although a different reference vector could have been chosen, notice that $\mathbf{r}_1(t)$ does not satisfy the lower bound on the norm. However, as discussed in Section 4.5, that does not affect the results and this vector was chosen on purpose to illustrate this situation. The filter parameter was chosen as $\mathbf{Q} = 0.1\mathbf{I}$, while

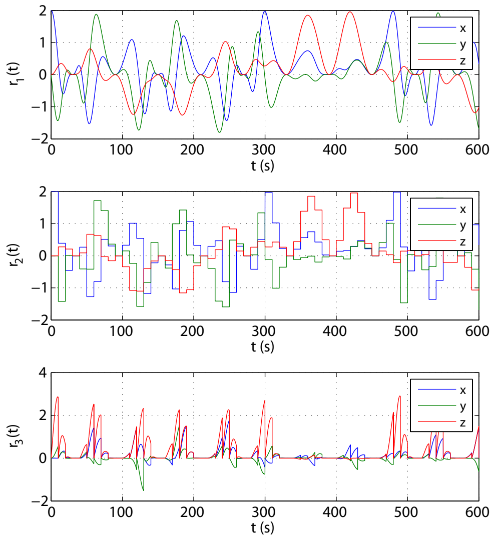

Fig. 1. Evolution of the reference vectors

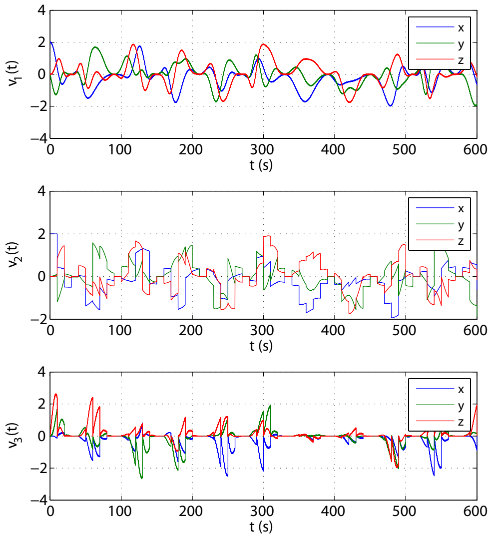

Fig. 2. Evolution of the body-fixed vectors

the initial rotation matrix estimate was chosen as $\hat{\mathbf{R}}(0) = \text{diag}(-1, -1, 1)$.

Sensor noise was considered on the angular velocity readings and the body-fixed vector observation. In particular, additive, zero-mean, white Gaussian noise was considered, with standard deviation of $1\,°/s$ for the angular velocity and 0.01 for the body-fixed vector observation. Notice that the specification for the angular velocity corresponds to a very low-grade sensor, while for the body-fixed vectors it corresponds to a standard deviation of 0.5% of the range. It should be emphasized that, when the norms of the body-fixed vectors are low, e.g. on the time interval $[20, 40]$ s, the noise of the vector observations is relatively very large. Therefore, the present specifications correspond to a realistic low-cost sensor suite.

The evolution of the Lyapunov function $V(t)$ is depicted in Fig. 3, where a logarithmic scale was employed for the y-axis. It is clear that the observer enters steady-state in less than 60 s, while the evolution of the error remains confined to a tight interval. Although it is not shown in the paper due to the lack of space, with the solution that resorts to the orthogonalization cycle the orthogonality error $\left\|\hat{\mathbf{R}}(t)\hat{\mathbf{R}}^T(t) - \mathbf{I}\right\|$ quickly reaches values around 10^{-4} with a single cycle and 10^{-8} with two orthogonalization cycles, which are very good considering the very noisy setup. Nevertheless, the solution provided in Theorem 7, instead of the use of orthogonalization steps, completely eliminates the orthogonality error, providing a solution on $SO(3)$. In order to evaluate the overall attitude performance, and for the purpose of performance evaluation only, an additional error variable is defined as $\tilde{\mathbf{R}}_a(t) = \mathbf{R}^T(t)\hat{\mathbf{R}}(t)$, which corresponds to the rotation matrix error. Using the Euler angle-axis representation for this new error variable,

$$\tilde{\mathbf{R}}_a(t) = \mathbf{I}\cos\left(\tilde{\theta}(t)\right) + \left[1 - \cos\left(\tilde{\theta}(t)\right)\right]\tilde{\mathbf{d}}(t)\tilde{\mathbf{d}}^T(t) \\ -\mathbf{S}\left(\tilde{\mathbf{d}}(t)\right)\sin\left(\tilde{\theta}(t)\right),$$

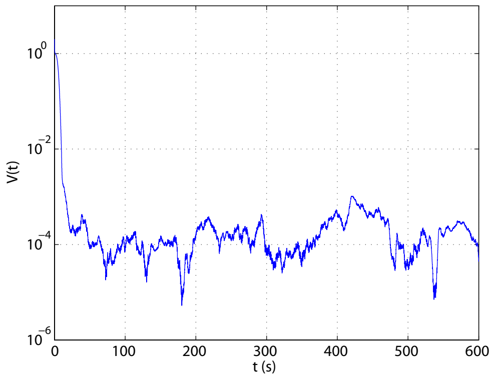

Fig. 3. Evolution of the Lyapunov function $V(t) = \frac{1}{2}\left\|\tilde{\boldsymbol{x}}(t)\right\|^2$

where $0 \leq \tilde{\theta}(t) \leq \pi$ and $\tilde{\mathbf{d}}(t) \in \mathbb{R}^3, \left\|\tilde{\mathbf{d}}(t)\right\| = 1$, are the angle and axis that represent the rotation error, the performance of the filter is easily identified from the evolution of $\tilde{\theta}$. Although it is not depicted here due to the lack of space, after the initial transients fade out, the angle error remains confined to a tight interval, in spite of the low-grade specifications of the sensors. The mean error is $0.68\,°$ which, considering the sensor suite specifications and the fact that one single reference vector is available in order to build the observer, consists in a very good and promising result.

Remark 3 *The theoretical stability analysis that is presented in this paper considers the nominal system dynamics, ignoring sensor noise. Nevertheless, the simulations results presented herein are extremely compelling regarding this issue. Indeed, the standard deviation of the noise that was considered for the vector observations reached very large relative values and the rate gyro noise is also very large, typical of the lowest grade rate gyros. Nevertheless, even with these noise characteristics, the observer achieved very interesting performances. The topic of stochastic stability of the proposed observer, considering a theoretical framework, will be subject of future research.*

6. Conclusions

This paper presented a novel set of attitude observers based on time-varying reference vectors. The origin of the error dynamics was shown to be a globally exponentially stable equilibrium point under appropriate persistent excitation conditions and, as the estimates of the observer converge only asymptotically to $SO(3)$, an explicit solution on $SO(3)$ was derived, whose error also converges exponentially fast to zero for all initial conditions. In addition, the distinct roles of the inertial and the corresponding body-fixed vectors on the observability of the system were also examined. Finally, simulation results were shown that illustrate excellent performance of the resulting attitude estimation solutions even with very low-grade sensor specifications.

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Appendix A. Auxiliary results

Lemma 8 *Let $\mathcal{A} = \{\mathbf{a}_i(t) \in \mathbb{R}^3, i = 1, \ldots, N\}$ be a persistently non-planar set of vectors. Then, there exist $\alpha > 0$ and $\delta > 0$ such that, for all $t \geq t_0$ and $\mathbf{d} \in \mathbb{R}^9$, $\|\mathbf{d}\| = 1$, it* *is true that*

$$\int_t^{t+\delta} \|\mathbf{b}_v(\mathbf{d},\,\sigma)\|\, d\sigma \geq \alpha,$$

where $\mathbf{d} = \begin{bmatrix} \mathbf{d}_1^T & \mathbf{d}_2^T & \mathbf{d}_3^T \end{bmatrix}^T$, $\mathbf{d}_i \in \mathbb{R}^3$, $i = 1,\,2,\,3$, *and*

$$\mathbf{b}_v(\mathbf{d}, t) = \begin{bmatrix} \begin{bmatrix} \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{d}_3 \end{bmatrix} \mathbf{a}_1(t) \\ \vdots \\ \begin{bmatrix} \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{d}_3 \end{bmatrix} \mathbf{a}_N(t) \end{bmatrix} \in \mathbb{R}^{3N}.$$

PROOF. The proof follows from simple norm inequalities. It is omitted due to the lack of space. □

Proposition 9 *Let* $\mathcal{R} = \{\mathbf{r}_1(t), \mathbf{r}_2(t)\}$, $\mathbf{r}_i(t) \in \mathbb{R}^3$, $i = 1, 2$, *be a persistently non-collinear set of vectors. Then, the augmented set of vectors* $\mathcal{R}_a := \mathcal{R} \cup \{\mathbf{r}_1(t) \times \mathbf{r}_2(t)\}$ *is persistently non-planar.*

PROOF. The proof follows by establishing that if $\mathcal{R}_a$ is not persistently non-planar, then $\mathcal{R}$ cannot be persistently non-collinear. To that purpose, suppose that $\mathcal{R}_a$ is not persistently non-planar. Then, for every $\alpha > 0$ and $\delta > 0$, it is possible to choose $t \geq t_0$ and $\mathbf{d} \in \mathbb{R}^3$, with $\|\mathbf{d}\| = 1$, such that

$$\int_t^{t+\delta} \|\mathbf{M}_r(\sigma)\,\mathbf{d}\|\, d\sigma < \alpha, \tag{A.1}$$

where

$$\mathbf{M}_r(\sigma) = \begin{bmatrix} \mathbf{r}_1^T(\sigma) \\ \mathbf{r}_2^T(\sigma) \\ [\mathbf{r}_1(\sigma)\times\mathbf{r}_2(\sigma)]^T \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

Applying simple norm inequalities in (A.1) gives

$$\begin{cases} \int_t^{t+\delta} \left\|\mathbf{r}_1^T(\sigma)\mathbf{d}\right\| d\sigma < \alpha \\ \int_t^{t+\delta} \left\|\mathbf{r}_2^T(\sigma)\mathbf{d}\right\| d\sigma < \alpha \\ \int_t^{t+\delta} \left\|[\mathbf{r}_1(\sigma)\times\mathbf{r}_2(\sigma)]^T\mathbf{d}\right\| d\sigma < \alpha \end{cases} \tag{A.2}$$

for all $\alpha > 0$ and $\delta > 0$. Now, let $\mathbf{d}^{\perp_1}$ be a unit vector orthogonal to $\mathbf{d}$, define $\mathbf{d}^{\perp_2} := \mathbf{d} \times \mathbf{d}^{\perp_1}$, and decompose $\mathbf{r}_1(\sigma)$ and $\mathbf{r}_2(\sigma)$ as

$$\begin{cases} \mathbf{r}_1(\sigma) = r_{10}(\sigma)\mathbf{d} + r_{11}(\sigma)\mathbf{d}^{\perp_1} + r_{12}(\sigma)\mathbf{d}^{\perp_2} \\ \mathbf{r}_2(\sigma) = r_{20}(\sigma)\mathbf{d} + r_{21}(\sigma)\mathbf{d}^{\perp_1} + r_{22}(\sigma)\mathbf{d}^{\perp_2} \end{cases}, \tag{A.3}$$

which is always possible as $\mathbf{d}$, $\mathbf{d}^{\perp_1}$, and $\mathbf{d}^{\perp_2}$ span $\mathbb{R}^3$. Substituting (A.3) in (A.2) immediately yields

$$\begin{cases} \int_t^{t+\delta} |r_{10}(\sigma)|\, d\sigma < \alpha \\ \int_t^{t+\delta} |r_{20}(\sigma)|\, d\sigma < \alpha \\ \int_t^{t+\delta} |r_{11}(\sigma)r_{22}(\sigma) - r_{12}(\sigma)r_{21}(\sigma)|\, d\sigma < \alpha \end{cases} \tag{A.4}$$

for all $\alpha > 0$ and $\delta > 0$. Now, notice that

$$\begin{aligned} \mathbf{r}_1(\sigma) \times \mathbf{r}_2(\sigma) = {} & [r_{11}(\sigma)r_{22}(\sigma) - r_{12}(\sigma)r_{21}(\sigma)]\,\mathbf{d} \\ & + [r_{12}(\sigma)r_{20}(\sigma) - r_{10}(\sigma)r_{22}(\sigma)]\,\mathbf{d}^{\perp_1} \\ & + [r_{10}(\sigma)r_{21}(\sigma) - r_{11}(\sigma)r_{20}(\sigma)]\,\mathbf{d}^{\perp_2}. \end{aligned} \tag{A.5}$$

Applying simple norm inequalities in (A.5), and as $\mathbf{d}$, $\mathbf{d}^{\perp_1}$, and $\mathbf{d}^{\perp_2}$ are unit vectors, yields

$$\begin{aligned} \|\mathbf{r}_1(\sigma) \times \mathbf{r}_2(\sigma)\| \leq {} & |r_{11}(\sigma)r_{22}(\sigma) - r_{12}(\sigma)r_{21}(\sigma)| \\ & + |r_{20}(\sigma)|\,|r_{12}(\sigma)| + |r_{10}(\sigma)|\,|r_{22}(\sigma)| \\ & + |r_{10}(\sigma)|\,|r_{21}(\sigma)| + |r_{20}(\sigma)|\,|r_{11}(\sigma)|\,. \end{aligned} \tag{A.6}$$

By definition, all vectors $\mathbf{r}_i(\sigma)$, $i = 1, \ldots, N$ are bounded. Let $c_1 := \sup_{\tau \geq t_0} |r_{ij}(\tau)|\,$, $i = 1, \ldots, N$, $j = 1, 2, 3$. Then, it can be concluded, from (A.6), that

$$\begin{aligned} \|\mathbf{r}_1(\sigma) \times \mathbf{r}_2(\sigma)\| \leq {} & |r_{11}(\sigma)r_{22}(\sigma) - r_{12}(\sigma)r_{21}(\sigma)| \\ & + 2c_1\,|r_{10}(\sigma)| + 2c_1\,|r_{20}(\sigma)| \end{aligned} \tag{A.7}$$

Integrating both sides of (A.7) and using (A.4) allows to conclude that

$$\int_t^{t+\delta} \|\mathbf{r}_1(\tau) \times \mathbf{r}_2(\tau)\|\, d\tau \leq \alpha\,[1 + 4c_1] \tag{A.8}$$

for all $\alpha > 0$ and $\delta > 0$. Then, it is clear that, for all $\alpha' > 0$ and $\delta' > 0$ it is possible to choose $t' \geq t_0$ such that

$$\int_{t'}^{t'+\delta'} \|\mathbf{r}_1(\sigma) \times \mathbf{r}_2(\sigma)\|\, d\sigma < \alpha',$$

just choose $t = t'$, $\delta = \delta'$, and $\alpha = \alpha'/(1 + 4c_1)$ in (A.8), which means that $\mathcal{R}$ is not persistently non-collinear. This concludes the proof, as it was shown that, if $\mathcal{R}_a$ is not persistently non-planar, then $\mathcal{R}$ is not persistently non-collinear. □

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